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Beliefs about Inflation and the Term Structure of Interest Rates[1]

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Abstract

We study how heterogeneity in preferences and difference in beliefs about expected inflation affect the nominal term structure when investors have external habit formation preferences in a pure exchange continuous time economy. Our benchmark common beliefs model is adapted from Chan and Kogan (2002) which broadly explains empirical features of asset prices. Our innovation is to introduce differences in beliefs about expected inflation. In particular, investors observe the path of exogenous consumption and the exogenous price level, but have different beliefs about expected inflation. Differences in beliefs about expected inflation alone impacts the equilibrium real pricing kernel generating a spill-over effect from the nominal to the real side of the economy. The equilibrium nominal stochastic discount factor is determined in closed form and the effects of aggregate risk aversion, difference in beliefs, and inflation on the nominal short rate and the nominal market price of risk are explored. To develop intuition concerning the role of different beliefs between investors and an econometrician, we consider simplifying cases where the term structures can be computed in closed-form. We demonstrate the importance of the econometrician's beliefs on perceived bond properties as well as how the nature of the difference in beliefs about inflation among investors is important in generating predictability in asset prices and creating spill-over effects to the real side of the economy.

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1 Introduction

The predictability of nominal bond returns and the high persistence of changes in their yields are important features of U.S. Treasury bonds. While there are many sophisticated reduced-form models that are successful in explaining some of these feature, the study of the economic mechanism behind these empirical stylized facts remains early in its development.[1] Moreover, the change in monetary policy in 1979, the high inflationary period in the 1970s, and the ability of the Fed over the last twenty years to keep inflation in check has led to changes in the dynamics of inflation. In particular, as discussed in Kroszner (2007)), it seems that the persistence in inflation has decreased over time and has affected the evolution of the term structure of interest rates. For instance, unconditional volatilities of changes in yields were decreasing in maturity for the pre Volcker-Greenspan period but are now hump-shaped, the hump occurring for two to three year maturities.

As Gürkaynak and Wright (2010) point out in their recent survey, inflation uncertainty seems to play an important role in potentially explaining the dynamics of nominal bond risk premiums. This is the departure point for our work where we explore the role that differences in beliefs about inflation dynamics among investors plays in determining properties of nominal bond prices. While other heterogeneous beliefs works have explored specific features of bond prices that can be explained, for example predictability in Xiong and Yan (2010), our work focuses on exploring the tensions introduced by heterogeneous beliefs in not just explaining predictability, but also explaining other asset pricing properties such as unconditional moments of nominal bond yields and spill-over effects of differences in inflation beliefs impacting real asset prices.

As with any heterogeneous beliefs model, a common problem is in linking investor beliefs to the true underlying economy. In fact, a common assumption is just to study the economy in a setting where one of the investors is assumed to have correct beliefs. Recent work by Piazzesi and Schneider (2011) argues that one potential source for predictability in long term bond excess returns is that investors' actual predictions while trading in bond markets are different from the predictions derived from econometricians by using historical bond data. We also accommodate for this difference in beliefs between investors and the econometrician allowing us to explore the impact of heterogeneous beliefs about inflation along two dimensions. First, heterogeneity of beliefs about inflation between

[1] See Singleton (2006) as well as recent surveys by Gürkaynak and Wright (2010) and Rudebusch (2010) including the references therein.

the investors in the economy allows us to capture the impact of speculative trade on bond properties. Second, heterogeneity of beliefs about inflation between the investors and the econometrician allows us to better understand the restrictions placed on investor beliefs to explain historical nominal bond price patterns.

To accomplish this, our work studies the equilibrium term structure of nominal interest rates in a pure exchange economy with heterogenous external habit formation preferences and inflation uncertainty. Our benchmark common beliefs model is adapted from Chan and Kogan (2002) which is already known to be able to broadly explain empirical features of asset prices. Our innovation is to introduce differences in beliefs about expected inflation. In particular, investors observe the path of exogenous nominal consumption and the exogenous price level, but have different beliefs about expected inflation. Differences in beliefs about expected inflation alone impacts the equilibrium real pricing kernel generating a spill-over effect from the nominal to the real side of the economy. The equilibrium nominal stochastic discount factor is determined in closed form and the effects of aggregate risk aversion, difference in beliefs, and inflation on the nominal short rate and the nominal market price of risk are explored. Moreover, we provide simple formulas for nominal bond prices, real bond prices, and the inflation risk premium that can be numerically evaluated.

To develop additional intuition concerning the role of both different beliefs between the investors and the econometrician and different beliefs among the investors, we consider simplifying cases that provide more detailed closed-form equilibrium solutions, in particular closed-form term structures in the class of quadratic Gaussian term structure models8. In these cases, we demonstrate the importance of the econometrician's beliefs on perceived bond properties as well as how the nature of the difference in beliefs about inflation among investors is important in generating predictability in asset prices and creating spill-over effects to the real side of the economy.

The paper proceeds as follows. Section 2 describes the basic economic setup. The equilibrium is defined and characterized in Section 3. Section 4 studies the role of differences in beliefs about inflation expectations on real and nominal term structure properties. Section 5 concludes.

2 The Economy

To study the equilibrium impact of heterogeneous beliefs about inflation, our economic environment is a continuous-time pure exchange economy with heterogeneous investors. The economy is similar to Chan and Kogan (2002) modified to incorporate an exogenous inflation process as well as heterogeneous beliefs about expected inflation. The economy has a finite horizon equal to T with a single perishable consumption good. Real prices are measured in units of the consumption good and nominal prices are measured in dollars. Uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$.

The exogenous real aggregate output $\epsilon(t)$ process follows a geometric Brownian motion with dynamics given by

$$d\epsilon(t) = \epsilon(t)\left[\mu_\epsilon\, dt + \sigma_\epsilon\, dz_\epsilon(t)\right], \qquad \epsilon(0) > 0, \tag{2.1}$$

where $z_\epsilon(t)$ is a one-dimensional Brownian motion that represents a real shock. The exogenous price level $\pi(t)$ in the economy is given by

$$d\pi(t) = \pi(t)\left[x(t)\, dt + \sigma_{\pi,\epsilon}\, dz_\epsilon(t) + \sigma_{\pi,\$}\, dz_\$(t)\right], \qquad \pi(0) = 1, \tag{2.2}$$

where $x(t)$ denotes expected inflation and $z_\$(t)$ is a one-dimensional Brownian motion that represents a nominal shock to the economy. The Brownian motions $z_\epsilon(t)$ and $z_\$(t)$ are locally uncorrelated. Since the price level also loads on $z_\epsilon(t)$, real aggregate output can be correlated with the price level.

Nominal aggregate output in the economy is denoted by $\epsilon_\$(t) \equiv \pi(t)\epsilon(t)$ with dynamics given by

$$d\epsilon_\$(t) = \epsilon_\$(t)\left[\mu_{\epsilon_\$}(t)\, dt + (\sigma_\epsilon + \sigma_{\pi,\epsilon})\, dz_\epsilon(t) + \sigma_{\pi,\$}\, dz_\$(t)\right], \qquad \epsilon_\$(0) > 0, \tag{2.3}$$

where $\mu_{\epsilon_\$}(t) \equiv \mu_\epsilon(t) + x(t) + \sigma_\epsilon \sigma_{\pi,\epsilon}$.

To close the exogenous processes requires making an assumption about the unobserved expected inflation process. We assume that it follows an Ornstein-Uhlenbeck process; however, the analysis of Section 3 still holds for other dynamics such as finite-state Markov processes as in Veronesi (1999, 2000). Specifically, we assume the dynamics of $x(t)$ are

$$dx(t) \;=\; \kappa\left(\bar{x} - x(t)\right) dt + \sigma_x\, dz_x(t), \quad x(0) \text{ given}, \tag{2.4}$$

where $x(0) \sim \mathcal{N}(\bar{x}(0), \sigma_x^2(0))$, $dz_x(t)\frac{d\epsilon(t)}{\sigma_\epsilon \epsilon(t)} = \rho_{x\epsilon} dt$, and $dz_x(t)\frac{d\pi(t)}{\sigma_\pi \pi(t)} = \rho_{x\pi} dt$.

2.1 Beliefs

Investors, as well as the econometrician, in the economy have heterogeneous beliefs about expected inflation because they only observe the continuous record of the price level $\pi(t)$. While both the econometrician and all investors agree that the evolution of the price level follows (2.2) respectively, they do not know the true expected inflation $x(t)$. Heterogeneous beliefs are modeled with investor-specific priors about these quantities as in for example Detemple and Murthy (1994), Basak (2000), and Basak (2005). Given $\pi(t)$ is observed continuously, all investors can perfectly estimate the volatilities in (2.2) by computing the quadratic variation of the process and the quadratic covariation with $\epsilon(t)$.

As with any economy when investors have different beliefs about economic fundamentals than the economy's objective beliefs, an important consideration is the linkage of the subjective beliefs of the investors to the objective beliefs of an econometrician as argued in works such as Cecchetti, Lam, and Mark (2000), Abel (2003), and Piazzesi and Schneider (2011). Here we explicitly model a set of beliefs for the econometrician distinct from our reference probabilities $\mathcal{P}$. This is driven by the desire to understand equilibrium properties through the lens of the econometrician's beliefs relative to the investors' beliefs.

For an econometrician denoted as $i = 0$ and investors denoted as $i = \{1, 2\}$, uncertainty in the economy is represented by the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t^{\epsilon,\pi}\}, \mathcal{P}^i\right)$ where $\mathcal{F}_t^{\epsilon,\pi}$ denotes the filtration generated by real output and the price level. When referring to both the econometrician and the investors, we use *agents*. Agent i's best estimate for expected inflation is

$$x^i(t) = \mathrm{E}^i[x(t) \mid \mathcal{F}_t^{\epsilon,\pi}], \quad i \in \{0, 1, 2\}, \tag{2.5}$$

where $\mathrm{E}^i[\cdot]$ denotes the expectation with respect to agent i's belief $\mathcal{P}^i$.

As long as the agents in the economy start with different priors about expected inflation, they will continue to disagree about these quantities over the life of the economy. From standard filtering theory as in Liptser and Shiryaev (1974a,b), investor i's innovation processes for $z_\epsilon(t)$ and $z_\$(t)$ are related to the reference probability $\mathcal{P}$ and the econometrician's beliefs $\mathcal{P}^0$ via

$$dz_\$^i(t) \quad = \quad dz_\$(t) + \frac{x(t) - x^i(t)}{\sigma_{\pi,\$}} dt = dz_\$^0(t) + \frac{x^0(t) - x^i(t)}{\sigma_{\pi,\$}} dt. \tag{2.6}$$

Given the dynamics of real output $\epsilon(t)$ are known, $z_\epsilon(t)$ is known by all agents. This implies there are no feedback effects on the real pricing kernel from disagreement about real quantities.

From equation (2.6), investors' innovation processes are related to the econometrician and to each other by

$$dz_\$^i(t) = dz_\$^0(t) + \Delta^i(t)dt, \quad i \in \{1,2\}, \tag{2.7}$$

$$dz_\$^2(t) = dz_\$^1(t) - \Delta(t)dt, \tag{2.8}$$

where the processes $\Delta^i(t)$ and $\Delta(t) = \Delta^1(t) - \Delta^2(t)$ represent disagreement processes with respect to the econometrician and across the two investors. They are defined as

$$\Delta^i(t) = \frac{x^0(t) - x^i(t)}{\sigma_{\pi,\$}} \quad \text{and} \quad \Delta(t) = \frac{x^2(t) - x^1(t)}{\sigma_{\pi,\$}}. \tag{2.9}$$

These processes summarize investors' differences in opinion about expected inflation. The disagreement is driven by initial priors and the paths of the realized real output and the price level.

We assume that the agents, including the econometrician, have different beliefs about expected inflation through two possible channels. Specifically, we focus on cases in which the econometrician and the investors differ with respect to (i) the long run mean of expected inflation $\bar{x}$, (ii) the speed of mean reversion of expected inflation κ, or (iii) both.

Since no participant in the economy, including the econometrician, perfectly observes expected inflation, they infer it by observing the path of the price level $\pi(t)$. Following standard filtering theory as in Liptser and Shiryaev (1974b), we can derive each market participant's dynamics for the estimator of expected inflation $x^i(t)$ where again we use $i = 1$, 2 to denote the two investors and $i = 0$ to denote the econometrician. This filtering problem is summarized as follows.

Proposition 1. *Under each agent's beliefs, the price level is*

$$d\pi(t) = \pi(t)\left[x^i(t)\,dt + \sigma_{\pi,\epsilon}\,dz_\epsilon(t) + \sigma_{\pi,\$}\,dz_\$^i(t)\right], \qquad \pi(0) = 1, \tag{2.10}$$

for $i \in \{0,1,2\}$.

Each agent's beliefs about the expected inflation rate follows the process:

$$dx^i(t) = \kappa^i \left(\bar{x}^i - x^i(t)\right) dt + \sigma^i_{x,\epsilon}\, dz_\epsilon(t) + \sigma^i_{x,\$}\, dz^i_\$(t), \quad x^i(0) \text{ given,} \tag{2.11}$$

for $i \in \{0,1,2\}$ where $x^i(0) \sim N\left(\bar{x}^i(0), \sigma^2_{x^i(0)}\right)$.

The volatilities $\sigma^i_{x,\epsilon}$ and $\sigma^i_{x,\$}$ for $i = 0, 1, 2$ are

$$\sigma^i_{x,\epsilon} = \sigma_{x,\epsilon} = \sigma_x\, \rho_{x\epsilon}, \tag{2.12}$$

$$\sigma^i_{x,\$} = \frac{\sigma_x}{\sqrt{1-\rho^2_{\epsilon\pi}}}\left(\rho_{x\pi} - \rho_{\epsilon\pi}\rho_{x\epsilon} + \frac{1}{\sigma_\pi \sigma_x} v_i\right) = \frac{\sigma_x}{\sqrt{1-\rho^2_{\epsilon\pi}}}\left(\rho_{x\pi} - \rho_{\epsilon\pi}\rho_{x\epsilon}\right) + \frac{v_i}{\sigma_{\pi,\$}}, \tag{2.13}$$

where v_i is agent i's estimation error.

Suppose the estimation error v^i is equal to its steady state value, i.e., it is a constant, given by

$$a\left(v^i\right)^2 + b^i v^i + c^i = 0, \tag{2.14}$$

with

$$a = -\frac{1}{\left(1-\rho^2_{\epsilon\pi}\right)\sigma^2_\pi}, \tag{2.15}$$

$$b^i = -2\kappa^i - \frac{2\sigma_x}{\sigma_\pi\left(1-\rho^2_{\epsilon\pi}\right)}\left(\rho_{\pi x} - \rho_{\epsilon\pi}\rho_{x\epsilon}\right), \tag{2.16}$$

$$c = \frac{\sigma^2_x}{1-\rho^2_{\epsilon\pi}}\left(1 - \rho^2_{\pi\epsilon} - \rho^2_{\pi x} - \rho^2_{x\epsilon} + 2\rho_{\pi\epsilon}\rho_{\pi x}\rho_{x\epsilon}\right). \tag{2.17}$$

The dynamics of disagreement between the first and the second investor are

$$\begin{aligned} d\Delta(t) = &\left(\frac{\kappa^2\bar{x}^2 - \kappa^1\bar{x}^1}{\sigma_{\pi,\$}} + \frac{\sigma^2_{x,\$} - \sigma^1_{x,\$}}{\sigma^2_{\pi,\$}} x^0(t) + \left[\frac{\kappa^1 - \kappa^2}{\sigma_{\pi,\$}} + \frac{\sigma^1_{x,\$} - \sigma^2_{x,\$}}{\sigma^2_{\pi,\$}}\right] x^1(t)\right. \\ &\left. - \left(\kappa^2 + \frac{\sigma^2_{x,\$}}{\sigma_{\pi,\$}}\right)\Delta(t)\right) dt + \frac{\sigma^2_{x,\$} - \sigma^1_{x,\$}}{\sigma_{\pi,\$}}\, dz^0_\$(t). \end{aligned} \tag{2.18}$$

Similarly, the dynamics of disagreement between the econometrician and investor i are

$$d\Delta^i(t) = \left(\frac{\kappa^0\bar{x}^0 - \kappa^i\bar{x}^i}{\sigma_{\pi,\$}} + \frac{\kappa^i - \kappa^0}{\sigma_{\pi,\$}} x^0(t) - \left(\kappa^i + \frac{\sigma^i_{x,\$}}{\sigma_{\pi,\$}}\right)\Delta^i(t)\right) dt + \frac{\sigma^0_{x,\$} - \sigma^i_{x,\$}}{\sigma_{\pi,\$}}\, dz^0_\$(t). \tag{2.19}$$

If there is only disagreement about the long run mean $\bar{x}$, then the disagreement $\Delta(t)$ and $\Delta_i(t)$ is deterministic.

2.2 Security Markets

Investors in the economy trade continuously in a real riskfree asset, a nominal money market account, and a security whose real return is locally perfectly correlated with real consumption growth. For simplicity, we will refer to this security as a "stock." This particular security structure is not crucial. We only require that the financial security structure be such that each investor can trade in a complete market.

The real risk-free asset is in zero-net supply and its real price is denoted by $B(t)$. The posited real price dynamics are

$$dB(t) = B(t)\, r(t)\, dt, \qquad B(0) = 1, \tag{2.20}$$

where $r(t)$ denotes the real riskfree rate to be determined in equilibrium. Investors observe the price of the real risk-free asset and hence know and agree on $r(t)$.

The nominal money market account is in zero-net supply and its nominal price is denoted by $P_\$(t)$. The posited nominal price dynamics are

$$dP_\$(t) = P_\$(t)\, r_\$(t)\, dt, \qquad dP_\$(0) = 1, \tag{2.21}$$

where $r_\$(t)$ denotes the nominal risk-free rate. Both investors agree on the nominal price of the nominal money market account and hence know and agree on the nominal risk-free rate $r_\$(t)$.

Applying Itô's lemma to the nominal price dynamics in equation (2.21) leads to the posited real price dynamics of the nominal money market account denoted $P(t) \equiv \frac{P_\$(t)}{\pi(t)}$:

$$dP(t) = P(t)\left[\mu_P(t)\, dt - \sigma_{\pi,\epsilon}\, dz_\epsilon(t) - \sigma_{\pi,\$}\, dz_\$(t)\right], \qquad \mu_P(t) \equiv r_\$(t) - x(t) + \sigma^2_{\pi,\epsilon} + \sigma^2_{\pi,\$}, \tag{2.22}$$

$$dP(t) = P(t)\left[\mu^i_P(t)\, dt - \sigma_{\pi,\epsilon}\, dz_\epsilon(t) - \sigma_{\pi,\$}\, dz^i_\$(t)\right], \qquad \mu^i_P(t) \equiv r_\$(t) - x^i(t) + \sigma^2_{\pi,\epsilon} + \sigma^2_{\pi,\$}, \tag{2.23}$$

where equation (2.23) shows the real price dynamics of the nominal money market under each investor's beliefs. Since both investors agree on the real price of the nominal money market $P(t)$, the investor-specific expected returns are linked through

$$\mu^1_P(t) - \mu^2_P(t) = \sigma_{\pi,\$}\, \Delta(t) = x^2(t) - x^1(t). \tag{2.24}$$

This difference in expected returns is solely driven by the disagreement about expected inflation.

Let $S(t)$ denote the real price of a "stock" in zero net supply which is locally perfectly correlated with real consumption growth and has a strictly positive volatility $\sigma_S(t) > 0$.[2] The posited price dynamics are

$$dS(t) = S(t)\left[\mu_S(t)\,dt + \sigma_{S,\epsilon}(t)\,dz_\epsilon(t)\right], \qquad S(0) = 1, \tag{2.25}$$

where $\mu_S(t)$ denotes the expected real rate of return on the stock under all agents' beliefs given there is no disagreement about $z_\epsilon(t)$.

The endogenous price system $(r(t), r_\$(t), \mu_S(t), \mu_P^1(t), \mu_P^2(t))$ is determined in a dynamically complete equilibrium. It is convenient to summarize the price system in terms of investor-specific real state price densities, or stochastic discount factors. Investor i's real state price density has dynamics

$$d\xi^i(t) = -\xi^i(t)\left[r(t)\,dt + \theta_\epsilon(t)\,dz_\epsilon(t) + \theta_\$^i(t)\,dz_\$^i(t)\right], \qquad \xi^i(0) = 1, \tag{2.26}$$

where

$$\theta_\epsilon(t) = \frac{\mu_S(t) - r(t)}{\sigma_S} \tag{2.27}$$

and investor i's perceived market prices of risk to the nominal shock $\theta_\$^i(t)$ is

$$\theta_\$^i(t) = -\frac{\mu_P^i(t) - r(t)}{\sigma_{\pi,\$}} - \frac{\sigma_{\pi,\epsilon}}{\sigma_{\pi,\$}}\theta_\epsilon(t). \tag{2.28}$$

Given investor 1 and 2 agree on the security prices as well as the real interest rate, the investor-specific market prices of risk are linked through the disagreement process:

$$\theta_\$^2(t) - \theta_\$^1(t) = \Delta(t). \tag{2.29}$$

2.3 Investor Preferences and Consumption-Portfolio Choice Problem

Investors share endowment risk by continuously trading in the security market. Investors may differ with respect to (strictly positive) initial wealth, beliefs, and preferences. The two types of investors have "catching up with the Joneses" preferences as in Abel (1990) and heterogenous risk aversion

[2] By specifying this security to be non-dividend paying and in zero net supply, the volatility $\sigma_S(t)$ can be taken as exogenous.

coefficients as in Chan and Kogan (2002):

$$U^i = \mathrm{E}^i \left[\int_0^T e^{-\rho t} u^i \left(\frac{c^i(t)}{X(t)} \right) dt \right], \qquad i = \{1, 2\}, \tag{2.30}$$

where ρ denotes the common subjective discount factor and $X(t)$ denotes the standard of living process. The standard of living is measured as a weighted "geometric sum" of past realizations of aggregate output

$$\log(X(t)) = \log(X(0))e^{-\delta t} + \delta \int_0^t e^{-\delta(t-a)} \log(\epsilon(a))\, da, \qquad \delta > 0, \tag{2.31}$$

where δ describes the dependence of $X(t)$ on the history of aggregate output.[3]

Define relative log output as $\omega(t) = \log(\epsilon(t)/X(t))$; it follows a mean reverting process

$$d\omega(t) = \delta(\bar{\omega}(t) - \omega(t))\, dt + \sigma_\epsilon\, dz_\epsilon(t), \tag{2.32}$$

with $\bar{\omega}(t) = (\mu_\epsilon(t) - \sigma_\epsilon^2/2)/\delta$. Conveniently, the pro-cyclical properties of relative output help to identify good and bad states of nature.

Investor i is endowed with a fraction of real aggregate output $\epsilon^i > 0$ where $\epsilon^1(t) + \epsilon^2(t) = \epsilon(t)$. The present value of investor's wealth is given by $W^i(0) = E^i \left[\int_0^T \xi^i(t)\epsilon^i(t) dt \right]$. He then chooses a nonnegative consumption process $c^i(t)$, and a portfolio process consisting of $\psi_B^i(t)$ shares in the real risk-free asset, $\psi_P^i(t)$ shares in the nominal money market account, and $\psi_S^i(t)$ shares in the stock.

Complete markets allow the use of standard martingale techniques (Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987)) to solve the consumption-portfolio problem of each investor. Optimal consumption processes $c^i(t)$ with supporting portfolio processes maximize the utility functions given in equation (2.30) subject to investor-specific static budget constraints $\mathrm{E}^i \left[\int_0^T \xi^i(t) c^i(t)\, dt \right] \leq W^i(0)$. The optimal consumption process is $\hat{c}^i(t) = X(t)\mathcal{I}\left(y^i e^{\rho t} X(t) \xi^i(t)\right)$, where $\mathcal{I}^i(\cdot)$ denotes the inverse function of $\partial u^i(a)/\partial a$ and where the Lagrange multipliers y^i are determined from the investor-specific static budget constraints.

[3] If δ is large, then shocks to relative output are transitory and hence the standard of living process resembles closely current output; i.e. $\omega(t) \approx 0$. If $\delta \approx 0$, then shocks to relative output are persistent and hence past aggregate output recieves high weight in the standard of living process.

3 Equilibrium

Financial security prices and optimal allocations are characterized by appealing to general equilibrium restrictions when investors disagree on the expected inflation rate.

Definition 1. *Given preferences, endowments, and beliefs, an equilibrium is a collection of allocations* $((c^1(t), \psi_P^1(t), \psi_S^1(t)), (c^2(t), \psi_P^2(t), \psi_S^2(t))$ *and a price system* $(r(t), \mu_S(t), \mu_P^1(t), \mu_P^2(t))$ *such that* $(c^i(t), \psi_P^i(t), \psi_S^i(t))$ *is an optimal solution to investor* i*'s consumption-portfolio problem given his perceived price processes, security prices are consistent across investors, and all markets clear for* $t \in [0, T]$. *Specifically,*

$$c^1(t) + c^2(t) = \epsilon(t), \qquad \psi_S^1(t) + \psi_S^2(t) = 0, \qquad \psi_P^1(t) + \psi_P^2(t) = 0.$$

The equilibrium can be constructed via a state-dependent representative agent as in Cuoco and He (1994) and Basak and Cuoco (1998) for example. The state-dependent representative agent at an arbitrary time t is constructed by

$$U(\epsilon(t), X(t), \lambda(t)) = \max_{\{c^1(t)+c^2(t)=\epsilon(t)\}} \left(u^1\left(\frac{c^1(t)}{X(t)}\right) + \lambda(t) u^2\left(\frac{c^2(t)}{X(t)}\right)\right). \tag{3.1}$$

Proposition 2 (Equilibrium). *If there exists an equilibrium, then the equilibrium consumption allocations are*

$$\begin{aligned} \hat{c}^1(\epsilon(t), X(t), \lambda(t)) &= X(t)\mathcal{I}^1\left(X(t)U_\epsilon\left(\epsilon(t), X(t), \lambda(t)\right)\right) = f\left(\epsilon(t), X(t), \lambda(t)\right)\epsilon(t), \\ \hat{c}^2(\epsilon(t), X(t), \lambda(t)) &= X(t)\mathcal{I}^2\left(\frac{X(t)}{\lambda(t)}U_\epsilon\left(\epsilon(t), X(t), \lambda(t)\right)\right) = \left(1 - f\left(\epsilon(t), X(t), \lambda(t)\right)\right)\epsilon(t), \end{aligned} \tag{3.2}$$

in which $\mathcal{I}^1(\cdot)$ *and* $\mathcal{I}^2(\cdot)$ *denote the inverse of* $(u^1)'(\cdot) = u_c^1$ *and* $(u^2)'(\cdot) = u_c^2$*, respectively, and where the sharing rule* $f(\cdot)$ *solves* $f\left(\epsilon(t), X(t), \lambda(t)\right) = \mathcal{I}^1\left(\lambda(t)u_c^2\left(e^{\omega(t)}\left(1 - f\left(\epsilon(t), X(t), \lambda(t)\right)\right)\right)\right)e^{-\omega(t)}$.

The investor's equilibrium state price densities are

$$\begin{aligned} \xi^1\left(\epsilon(t), X(t), \lambda(t)\right) &= e^{-\rho t}\frac{U_\epsilon\left(\epsilon(t), X(t), \lambda(t)\right)}{U_\epsilon\left(\epsilon(0), X(0), \lambda(0)\right)} \\ &= e^{-\rho t + \omega(t) - \omega(0)}\frac{u_c^1\left(e^{\omega(t)}s\left(\epsilon(t), X(t), \lambda(t)\right)\right)\epsilon(0)}{u_c^1\left(e^{\omega(0)}s\left(\epsilon(0), X(0), \lambda(0)\right)\right)\epsilon(t)}, \end{aligned} \tag{3.3}$$

$$\xi^2\left(\epsilon(t), X(t), \lambda(t)\right) = \xi^1\left(\epsilon(t), X(t), \lambda(t)\right)\frac{\lambda(0)}{\lambda(t)}, \tag{3.4}$$

where the stochastic welfare weight $\lambda(t)$ has dynamics

$$d\lambda(t) = \lambda(t)\,\Delta(t) dz^1_{\$}(t), \tag{3.5}$$

and $\lambda(0)$ solves either investor's static budget constraint.

Given we are also interested in computing equilibrium quantities from the perspective of the econometrician, the real state price density from the perspective of the econometrician can be expressed as

$$\xi^0(t) = \lambda_i(t)\xi^i(t), \quad \text{where } \frac{d\lambda_{i(t)}}{\lambda_i(t)} = -\Delta^i(t) dz^0_{\$}(t). \tag{3.6}$$

In particular, $\lambda(t) = \frac{\lambda_2(t)}{\lambda_1(t)}$.

From the equilibrium construction of the real state price density for each investor, the equilibrium interest rate and market prices of risk can be computed highlighting the impact of belief heterogeneity about expected inflation on real prices.

Proposition 3. *The market prices of risk for each investor $i \in \{1, 2\}$ are as follows:*

$$\theta_\epsilon(t) = \mathcal{A}(t)\epsilon(t)\sigma_\epsilon, \qquad \theta^1_{\$}(t) = -\frac{\mathcal{A}(t)}{\mathcal{A}_2(t)}\Delta(t), \qquad \theta^2_{\$}(t) = \frac{\mathcal{A}(t)}{\mathcal{A}_1(t)}\Delta(t), \tag{3.7}$$

where $\mathcal{A}_i(t)$, $i \in \{1, 2\}$ is the absolute risk aversion coefficient for each investor and $\mathcal{A}(t)$ is the absolution risk aversion coefficient for the state-dependent representative agent where $\mathcal{A}(t) \equiv \frac{1}{\frac{1}{\mathcal{A}_1(t)} + \frac{1}{\mathcal{A}_2(t)}}$.

The real interest rate is given by

$$\begin{aligned} r(t) = \rho + \delta\omega(t) + \mathcal{A}(t)\epsilon(t)\left(\mu^1_\epsilon(t) - \delta\omega(t)\right) \\ - \frac{1}{2}\mathcal{A}(t)\mathcal{P}_1(t)\left[\left(\frac{\theta_\epsilon(t)}{\mathcal{A}_1(t)}\right)^2 + \left(\frac{\theta^1_{\$}(t)}{\mathcal{A}_1(t)}\right)^2\right] - \frac{1}{2}\mathcal{A}(t)\mathcal{P}_2(t)\left[\left(\frac{\theta_\epsilon(t)}{\mathcal{A}_2(t)}\right)^2 + \left(\frac{\theta^2_{\$}(t)}{\mathcal{A}_2(t)}\right)^2\right] \\ - \frac{\mathcal{A}(t)}{\mathcal{A}_2(t)}\left(\theta^1_{\$}(t) - \theta^2_{\$}(t)\right)\theta^2_{\$}(t), \end{aligned} \tag{3.8}$$

where $\mathcal{P}_i(t)$, $i \in \{1, 2\}$ is the absolute prudence coefficient for each investor.

Proposition 3 highlights the impact of speculative trade on real prices in the economy. Although the investors in the economy do not disagree about any real quantities, disagreement about expected inflation, a nominal quantity, induces a spillover effect on the real side of the economy as the nominal shock $z_\$(t)$ is now priced through $\theta^i(t)$. This mechanism, that heterogeneous beliefs about a nominal quantity can induce nominal risks to be priced on the real side of the economy, is distinct from New-Keynesian models such as Clarida, Gali, and Gertler (1999) where mechanisms such as sticky prices are imposed so that the nominal side of the economy impacts the real side of the economy.

We now use the state price densities determined in Proposition 2 to determine real and nominal bond prices. All bonds considered in this paper are default-free zero-coupon bonds. A real bond pays one unit of the consumption good at its maturity and a nominal bond pays one unit of currency at its maturity. Real bonds and nominal bonds are in zero net supply.

Let $B(t;T')$ denote the real and $B_\$(t;T') = B(t;T')\pi(t)$ the nominal price of a real (inflation-protected) bond maturing at T'. The real price of a real bond with maturity T' is

$$B(t;T') = \mathrm{E}_t^i\left[\frac{\xi^i(T')}{\xi^i(t)}\right]. \tag{3.9}$$

Let $P(t;T')$ denote the real and $P_\$(t;T') = P(t;T)\pi(t)$ the nominal price of a nominal bond maturing at T'. The nominal price of a nominal bond with maturity T' is

$$P_\$(t;T') = \mathrm{E}_t^i\left[\frac{\xi^i(T')}{\xi^i(t)}\frac{\pi(t)}{\pi(T')}\right]. \tag{3.10}$$

Analogously, define the log-yields at time t of a real and nominal zero-coupon bond with τ years to maturity as $y_B^{(\tau)}(t) = -\frac{1}{\tau}\log\left(B(t;t+\tau)\right)$ and $y_{P_\$}^{(\tau)}(t) = -\frac{1}{\tau}\log\left(P_\$(t;t+\tau)\right)$ respectively. We provide closed form solutions for real and nominal bond prices in the next sections.

4 The Impact of Different Beliefs Across Investors and the Econometrician with Closed-Form Bond Prices

To gain insights on the impact of heterogenous belief about expected inflation on bond prices, it is useful to consider a setting where all bond prices can be computed in closed-form. To accomplish this, we assume that both habit-utility investors in the economy are endowed with a common integer risk aversion γ. This assumption allows us to construct exact expansions of bond prices in artificial economies. Again, while the exogenous price level is observable, expected inflation $x(t)$, which follows an Ornstein-Uhlenbeck process from equation (2.4), is not. Here, the investors have subjective beliefs about expected inflation in the economy. We assume that the econometrician has different beliefs about expected inflation through two possible channels. Specifically, we focus on cases in which the econometrician and the investors differ with respect to (i) the long run mean of expected inflation $\bar{x}$, (ii) the speed of mean reversion of expected inflation κ, or (iii) both.

Following the analysis from Sections 2 and 3, the consumption sharing rule is given by

$$f(t) = c_1(t)/\epsilon(t) = \frac{1}{1+\lambda(t)^{\frac{1}{\gamma}}} \qquad \text{with} \qquad \lambda(t) = \frac{\lambda_2(t)}{\lambda_1(t)}, \tag{4.1}$$

leading to a real state price density as perceived by the econometrician given by

$$\xi^0(t) = \lambda_1(t)\left(1+\lambda(t)^{\frac{1}{\gamma}}\right)^{\gamma} e^{-\rho t}\epsilon(t)^{-\gamma}e^{(1-\gamma)\omega(t)}. \tag{4.2}$$

Under the econometrician's beliefs, the equilibrium real interest rate and market prices of risk are summarized as follows.

Proposition 4. *The dynamics of the econometrician's state price density $\xi^0(t)$ are*

$$d\xi^0(t) = -\xi^0(t)\left[r(t)dt + \theta_\epsilon(t)dz_\epsilon + \theta^0_\$(t)dz^0_\$(t)\right], \tag{4.3}$$

where the real market prices of risk for the econometrician are as follows:

$$\theta_\epsilon(t) = \gamma\sigma_\epsilon, \quad \theta^0_\$(t) = \Delta_1(t) + (1-f(t))\left(\Delta_2(t) - \Delta_1(t)\right), \tag{4.4}$$

and the real interest rate is given by

$$\begin{aligned} r(t) &= \rho + \mu_\epsilon - \frac{1}{2}(\gamma^2+1)\sigma_\epsilon^2 + \delta(\gamma-1)(\bar{\omega}-\omega(t)) \\ &+ \frac{1}{2}\left(1-\frac{1}{\gamma}\right) f(t)(1-f(t))(\Delta_2(t)-\Delta_1(t))^2. \end{aligned} \tag{4.5}$$

The real equilibrium price system as viewed by the econometrician in Proposition 4 cleanly demonstrates a spillover effect that heterogeneous beliefs about nominal quantities through inflation dynamics can have on the real side of the economy. In particular, once heterogenous beliefs are introduced, the real equilibrium interest rate $r(t)$ is now stochastic and is driven by the difference in beliefs between the two investors captured through $\Delta_2(t) - \Delta_1(t)$.

Additionally, the price level shock as perceived by the econometrician, $z_\$^0(t)$, is also priced as can be seen in the expression for $\theta_\$^0(t)$. This manifests itself through two channels. First, if the investors have different beliefs through $\Delta_2(t) - \Delta_1(t) \neq 0$, then $\theta_\$^0(t) \neq 0$. Here the pricing of the nominal shock is being driven by speculative trade between the investors through their disagreement. Second, the econometrician can perceive the nominal shock to be priced in the real pricing kernel when both investors agree ($\Delta_2(t) - \Delta_1(t) = 0$), but the econometrician disagrees with the common investor belief through $\Delta_1(t) \neq 0$. In particular, this points out how predictability in real asset prices can even arise through differences in belief about a nominal quantity.

4.1 Closed-Form Bond Prices

It is possible to compute bond prices in closed-form in this setting to better explore the role that heterogeneous beliefs play on real and nominal bond prices. To accomplish this, we decompose the real state price density through the following decomposition.

Proposition 5. *Assuming that γ is an integer, we can decompose the real state price density as*

$$\frac{\xi^0(t)}{\xi^0(0)} = \sum_{k=0}^{\gamma} w_k(t) \frac{\xi_k^0(t)}{\xi_k^0(0)}, \tag{4.6}$$

where $\xi_k^0(t)$ can be interpreted as a real state price density in a fictitious economy given by

$$\xi_k^0(t) = e^{-\rho t}\lambda_1(t)\lambda(t)^{\frac{k}{\gamma}}\epsilon(t)^{-\gamma}e^{(1-\gamma)\omega(t)}. \tag{4.7}$$

The dynamics of $\xi_k^0(t)$ are

$$\frac{d\xi_k^0(t)}{\xi_k^0(t)} = -r_k(t)dt - \theta_{k,\epsilon}(t)dz_\epsilon(t) - \theta_{k,\$}^0(t)dz_\$^0(t)(t), \tag{4.8}$$

where

$$\begin{aligned} r_k(t) &= \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2 - \delta(\gamma-1)\omega(t) \\ &\quad - \frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma}-1\right)\frac{1}{\sigma_{\pi,\$}^2}(x^1(t)-x^2(t))^2 \end{aligned} \tag{4.9}$$

$$\theta_{k,\epsilon}(t) = \gamma\sigma_\epsilon, \tag{4.10}$$

and

$$\begin{aligned} \theta_{k,\$}^0(t) &= \frac{1}{\sigma_{\pi,\$}}\left(x(t)-x^1(t)\right) + \frac{k}{\gamma}\frac{1}{\sigma_{\pi,\$}}\left(x^1(t)-x^2(t)\right) \\ &= \frac{1}{\sigma_{\pi,\$}}x(t) - \frac{x^1(t)}{\sigma_{\pi,\$}}\left(1-\frac{k}{\gamma}\right) - \frac{k}{\gamma}\frac{x^2(t)}{\sigma_{\pi,\$}}. \end{aligned} \tag{4.11}$$

The quantity $w_k(t)$ denotes the weight placed on $\xi_k(t)$ and is given by

$$w_k(t) = \begin{pmatrix} \gamma \\ k \end{pmatrix} \frac{\lambda(t)^{\frac{k}{\gamma}}}{\left(1+\lambda(t)^{\frac{1}{\gamma}}\right)^{\gamma}} = \begin{pmatrix} \gamma \\ k \end{pmatrix} f(t)^{\gamma-k}(1-f(t))^k \tag{4.12}$$

with $\sum_{k=0}^{\gamma} w_k(t) = 1$.

Likewise, we can also decompose the nominal state price density when γ is an integer as

$$\frac{\xi_\$^0(t)}{\xi_\$^0(0)} = \sum_{k=0}^{\gamma} w_k(t)\frac{\xi_{k\$}^0(t)}{\xi_{k\$}^0(0)}, \tag{4.13}$$

where $\xi_\$^0(t) = \frac{\xi^0(t)}{\pi(t)}$ and $\xi_{k\$}^0(t) = \frac{k\xi^0(t)}{\pi(t)}$. The dynamics of $\xi_{k\$}^0(t)$ are summarized in the following corollary.

Corollary 1. *The nominal stochastic discount factor in artificial economy k is defined as $\xi_{k\$}^0(t) = \xi_k^0(t)/\pi(t)$. We then have*

$$\frac{d\xi_{k\$}^0(t)}{\xi_{k\$}^0(t)} = -r_{k\$}(t)dt - \theta_{k\$,\epsilon}dz_\epsilon(t) - \theta_{k\$,\$}^0 dz_\$^0(t), \tag{4.14}$$

where

$$r_{k\$}(t) = r_k(t) + x^0(t) - \theta_{k,\epsilon}(t)\rho_{\epsilon\pi}\sigma_\epsilon - \theta^0_{k,\$}(t)\sigma_{\pi,\$} - \sigma^2_\pi, \tag{4.15}$$

$$\theta_{k\$,\epsilon}(t) = \gamma\sigma_\epsilon + \rho_{\epsilon\pi}\sigma_\epsilon, \tag{4.16}$$

and

$$\theta^0_{k\$,\$}(t) = \frac{1}{\sigma_{\pi,\$}}x^0(t) - \frac{x^1(t)}{\sigma_{\pi,\$}}\left(1 - \frac{k}{\gamma}\right) - \frac{k}{\gamma}\frac{x^2(t)}{\sigma_{\pi,\$}} + \sigma_{\pi,\$}. \tag{4.17}$$

These real and nominal state price density decompositions allow us to write the prices of bonds also as bond price decompositions in the k fictitious economies.

The real price of a real zero-coupon bond is therefore

$$B(t;T') = \sum_{k=0}^{\gamma} w_k(t)B_k(t;T'), \tag{4.18}$$

where $B_k(t;T')$ denotes the real price of a real bond in artificial economy k given by

$$B_k(t;T') = E^0_t\left[\frac{\xi^0_k(T')}{\xi^0_k(t)}\right] = E^0\left[\frac{\xi^0_k(T')}{\xi^0_k(t)} \mid \omega(t) = \omega, x^0(t) = x^0, x^1(t) = x^1, x^2(t) = x^2\right]. \tag{4.19}$$

Likewise, the nominal price of a nominal zero-coupon bond is therefore

$$P_\$(t;T') = \sum_{k=0}^{\gamma} w_k(t)P_{\$k}(t;T'), \tag{4.20}$$

where $P_{\$k}(t;T')$ denotes the nominal price of a nominal bond in artificial economy k given by

$$P_{\$k}(t;T') = E^0_t\left[\frac{\xi^0_{k\$}(T')}{\xi^0_{k\$}(t)}\right] = E^0\left[\frac{\xi^0_{k\$}(T')}{\xi^0_{k\$}(t)} \mid \omega(t) = \omega, x^0(t) = x^0, x^1(t) = x^1, x^2(t) = x^2\right]. \tag{4.21}$$

Given the structure of the artificial economies, we now show that the real and nominal term structures in them are in the class of quadratic Gaussian term structure models as discussed in Ahn, Dittmar, and Gallant (2002) for example. To show this mapping, we adopt largely the same notation as Ahn, Dittmar, and Gallant (2002) for the state vector $Y(t)$ in the economy. In particular, define $Y(t) = (x^0(t), x^1(t), x^2(t), \omega(t))'$. The dynamics of $Y(t)$ are

$$dY(t) = (\mu + \xi Y(t))\ dt + \Sigma dZ_2(t), \tag{4.22}$$

where

$$\mu = (\kappa_0 \bar{x}_0, \kappa_1 \bar{x}_1, \kappa_2 \bar{x}_2, \delta\bar{\omega})' \in \mathcal{R}^4, \tag{4.23}$$

$$\xi = \begin{pmatrix} -\kappa_0 & 0 & 0 & 0 \\ \frac{\sigma^1_{x,\$}}{\sigma_{\pi,\$}} & -\left(\kappa^1 + \frac{\sigma^1_{x,\$}}{\sigma_{\pi,\$}}\right) & 0 & 0 \\ \frac{\sigma^2_{x,\$}}{\sigma_{\pi,\$}} & 0 & -\left(\kappa^2 + \frac{\sigma^2_{x,\$}}{\sigma_{\pi,\$}}\right) & 0 \\ 0 & 0 & 0 & -\delta \end{pmatrix} \in \mathcal{R}^{4\times 4}, \tag{4.24}$$

$$\Sigma = \begin{pmatrix} \sigma^0_{x,\epsilon,0} & \sigma^0_{x,\$} \\ \sigma^1_{x,\epsilon,1} & \sigma^1_{x,\$} \\ \sigma^2_{x,\epsilon,2} & \sigma^2_{x,\$} \\ \sigma_\epsilon & 0 \end{pmatrix} \in \mathcal{R}^{4\times 2}, \tag{4.25}$$

and

$$Z_2(t) = \left(z_{\epsilon,0}(t), z^0_{\$}(t)\right)' \in \mathcal{R}^2. \tag{4.26}$$

The volatilities $\sigma^i_{x,\epsilon}$ and $\sigma^i_{x,\$}$ for $i = 0, 1, 2$ are

$$\sigma^i_{x,\epsilon} = \sigma_{x,\epsilon} = \sigma_x \, \rho_{x\epsilon}, \tag{4.27}$$

$$\sigma^i_{x,\$} = \frac{\sigma_x}{\sqrt{1-\rho^2_{\epsilon\pi}}} \left(\rho_{x\pi} - \rho_{\epsilon\pi}\rho_{x\epsilon} + \frac{1}{\sigma_\pi \sigma_x} v_i\right) = \frac{\sigma_x}{\sqrt{1-\rho^2_{\epsilon\pi}}} \left(\rho_{x\pi} - \rho_{\epsilon\pi}\rho_{x\epsilon}\right) + \frac{v_i}{\sigma_{\pi,\$}}, \tag{4.28}$$

and the estimation error v_i solves

$$a\,(v_i)^2 + b_i v_i + c = 0. \tag{4.29}$$

with

$$a = -\frac{1}{\left(1-\rho^2_{\epsilon\pi}\right)\sigma^2_\pi}, \tag{4.30}$$

$$b^i = -2\kappa^i - \frac{2\sigma_x}{\sigma_\pi\left(1-\rho^2_{\epsilon\pi}\right)}\left(\rho_{\pi x} - \rho_{\epsilon\pi}\rho_{x\epsilon}\right), \tag{4.31}$$

$$c = \frac{\sigma^2_x}{1-\rho^2_{\epsilon\pi}}\left(1 - \rho^2_{\pi\epsilon} - \rho^2_{\pi x} - \rho^2_{x\epsilon} + 2\rho_{\pi\epsilon}\rho_{\pi x}\rho_{x\epsilon}\right). \tag{4.32}$$

We determine the real bond price in the artificial economy k in the next proposition:

Proposition 6. *The real bond price in the artificial economy k is an exponential quadratic function* *of the state vector*

$$B_k(t;T') = exp\left\{A_k(T'-t) + B_k(\tau)'Y(T'-t) + Y(t)'C_k(T'-t)Y(t)\right\}, \tag{4.33}$$

where $A_k(T'-t)$, $B_k(\tau)$, *and* $C_k(T'-t)$ *are the solutions of the ode's given in the Appendix and*

$$\eta_{0,k} = -\left(\gamma\sigma_\epsilon, 0\right)' \tag{4.34}$$

$$\eta_{Y1,k} = 0_4 \tag{4.35}$$

$$\eta_{Y2,k} = -\frac{1}{\sigma_{\pi,\$}}\left(1, -\left(1-\frac{k}{\gamma}\right), -\frac{k}{\gamma}, 0\right)' \tag{4.36}$$

$$\alpha_k = \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma\left(\gamma+1\right)\sigma_\epsilon^2 \tag{4.37}$$

$$\beta_k = (0,0,0,\delta(1-\gamma))' \tag{4.38}$$

$$\Psi_k = -\frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma}-1\right)\frac{1}{\sigma_{\pi,\$}^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.39}$$

The matrix Ψ_k is positive semidefinite because $k/\gamma \leq 1$. Note Ψ_k is singular and that $\psi_K = 0_{4\times 4}$ if $k = 0$ or $k = \gamma$.

We determine the nominal bond price in the artificial economy k in the next proposition:

Proposition 7. *The nominal bond price in the artificial economy* k *is an exponential quadratic function of the state vector*

$$P_{k\$}(t;T') = exp\left\{A_{k\$}(T'-t) + B_{k\$}(\tau)'Y(T'-t) + Y(t)'C_{k\$}(T'-t)Y(t)\right\}, \tag{4.40}$$

where $A_{k\$}(T'-t)$, $B_{k\$}(\tau)$, and $C_{k\$}(T'-t)$ are the solutions of the ode's given in the Appendix and

$$\eta_{0,k\$} = -\left(\gamma\sigma_\epsilon + \rho_{\epsilon\pi}\sigma_\epsilon, \sigma_{\pi,\$}\right)' \tag{4.41}$$

$$\eta_{Y1,k\$} = 0_4 \tag{4.42}$$

$$\eta_{Y2,k\$} = -\frac{1}{\sigma_{\pi,\$}}\left(1, -\left(1-\frac{k}{\gamma}\right), -\frac{k}{\gamma}, 0\right)' \tag{4.43}$$

$$\alpha_{k\$} = \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2 - \gamma\rho_{\epsilon\pi}\sigma_\epsilon\sigma_\pi - \sigma_\pi^2 \tag{4.44}$$

$$\beta_{k\$} = \left(0, 1-\frac{k}{\gamma}, \frac{k}{\gamma}, \delta(1-\gamma)\right)' \tag{4.45}$$

$$\Psi_{k\$} = -\frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma}-1\right)\frac{1}{\sigma_{\pi,\$}^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.46}$$

The matrix $\Psi_{k\$}$ is positive semidefinite because $k/\gamma \leq 1$. Note that $\psi_{k\$}$ is singular and $\psi_{k\$} = 0_{4\times4}$ if $k=0$ or $k=\gamma$.

4.2 The Impact of Different Beliefs between a Representative Investor and an Econometrician

To isolate the role played by the econometrician having different beliefs than the investors in the economy, we explore a simplified version of the economy here. In particular, the economy is populated by a single representative investor with CRRA preferences and hence no habit effect. In particular, the sharing rule collapses to $f(t) = 1$.

This leads to a special case of our earlier equilibrium price system summarized as follows.

Corollary 2. *If the economy is populated by a single representative investor with CRRA preferences and a relative risk aversion coefficient of γ, equilibrium prices are summarized as follows.*

The real state price density $\xi^0(t) = e^{-\rho t}u'(\epsilon(t))$ has dynamics

$$\frac{d\xi^0(t)}{\xi^0(t)} = -r\,dt - \gamma\sigma_\epsilon\,dz_\epsilon(t),$$

where only shocks to real output are priced and the real risk-free rate is constant:

$$r = \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2.$$

Given expected inflation follows a mean-reverting Ornstein-Uhlenbeck process, the nominal state price density $\xi_\$^0(t) = \xi^0(t)/\pi(t)$ *with dynamics*

$$\frac{d\xi_\$(t)}{\xi_\$(t)} = -r_\$(t)\,dt - (\kappa_\epsilon + \sigma_\pi\rho_{\epsilon\pi})\;dz_\epsilon(t) - \sigma_\pi\sqrt{1-\rho_{\epsilon\pi}^2}\,dz_\$^0(t).$$

where the nominal price of risk is constant and the nominal short rate is stochastic:

$$r_\$(t) = r + x^0(t) - \gamma\sigma_\epsilon\sigma_\pi\rho_{\epsilon\pi} - \sigma_\pi^2.$$

With this price system, nominal bond prices follow a one factor completely affine Vasicek model where properties of bond yields are summarized in the following corollary.

Corollary 3. *Nominal bond yields are given by*

$$y_{P_\$}^{(\tau)}(t) = A_{P_\$}(\tau) + B_{P_\$}(\tau)r_\$(t) \tag{4.47}$$

with

$$B_{P_\$}(\tau) = \frac{1}{\kappa^i\tau}\left(1 - e^{-\kappa^i\tau}\right)$$

and

$$A_{P_\$}(\tau) = \left(\theta_\$ - \frac{(\sigma_x^i)^2}{2(\kappa^i)^2}\right)\left(1 - B_{P_\$}(\tau)\right) + \frac{(\sigma_x^i)^2}{4\kappa^i}\tau B_{P_\$}(\tau)^2$$

In particular, this allows us to compute nominal bond properties in closed-form comparing the econometrician's beliefs with that of the representative investor's beliefs. Average nominal bond yields are given by

$$\mathrm{E}^0\left[y_{P_\$}^{(\tau)}(t)\right] = \mathrm{E}^i\left[y_{P_\$}^{(\tau)}(t)\right] + B_{P_\$}(\tau)\left(\bar{x}^0 - \bar{x}^i\right).$$

The yield variances are

$$\mathrm{Var}^0\left[y_{P_\$}^{(\tau)}(t)\right] = \mathrm{Var}^i\left[y_{P_\$}^{(\tau)}(t)\right] + B_{P_\$}(\tau)^2\left(\frac{\left(\sigma_x^0\right)^2}{2\kappa^0} - \frac{(\sigma_x^i)^2}{2\kappa^i}\right).$$

Finally, the conditional risk premiums are

$$\mathrm{E}_t^0\left[\mathrm{rx}_{P_\$}^{(\tau)}(t,t+h)\right] = \mathrm{E}_t^i\left[\mathrm{rx}_{P_\$}^{(\tau)}(t,t+h)\right] + (\tau-h)\, B_{P_\$}(\tau-h)\left(\mathrm{E}_t^i - \mathrm{E}_t^0\right)\left[\hat{x}^i(t+h) - \hat{x}^0(t+h)\right].$$

The risk premiums are constant if the econometrician's beliefs coincide with the belief of the investor or if they only disagree about the long run mean of inflation. When they disagree about the mean reversion of expected inflation, then bond risk premiums are stochastic.

4.2.1 Numerical Examples with a CRRA Representative Investor

We now turn to exploring some numerical examples to better understand how properties of the yield curve are impacted when the beliefs of the representative investor and the econometrician do not coincide. The parameters used throughout this example are given in Table 1.

Figures 1 and 2 shows average yields and yield volatility as a function of maturity in the data and as predicted by the model. Not surprisingly, given our Vasicek setting the average yield curve in the model is downward sloping.

Our focus however is to explore properties of the yield curve when the beliefs of the representative investor an the econometrician do not coincide. We first focus on expectations about average yields when the representative investor and the econometrician disagree about the long run mean of inflation. Figure 3 demonstrates that even if in the model average yields are too high and downward sloping, an econometrician perceives lower average yields and an upward sloping yield curve if he believes that the long run mean of expected inflation is lower than that of the representative investor. The figure shows the unconditional mean of nominal bond yields as a function of the econometrician's long run mean of expected inflation $\bar{x}^0$. The yield curve is upward sloping in the data (Fama-Bliss data set). This is shown by the three dashed lines. The average yield curve from the perspective of the econometrician is shown by the three solid lines. The green line represents the long run mean of the representative investor (RI) in the model (yields are determined with respect to his belief). The yield curve is flat (slightly downward sloping) from the perspective of the RI as the intersection of the green line with

Table 1: Parameter Choice for CRRA Representative Investor Example.

Parameter	Description	Value
Preferences		
ρ	Time Preference Parameter	-1%
γ	Risk Aversion	3
Consumption		
μ_ϵ	Expected Consumption Growth	1.72%
σ_ϵ	Volatility of Consumption	3.32%
Inflation		
σ_π	Inflation Volatility	1.3%
$\bar{x}$	Long Run Mean of Expected Inflation	3%
κ	Mean Reversion of Expected Inflation	0.35
σ_x	Volatility of Expected Inflation	1.4%
$\rho_{\epsilon\pi}$	ρ of Realized Inflation & Real Consumption Growth	-0.2
$\rho_{\pi x}$	ρ of Realized and Expected Inflation	0
$\rho_{x\epsilon}$	ρ of Expected Inflation & Real Consumption Growth	0

the other three solid lines shows. If the econometrician's long run mean is lower than that of the RI, then it is possible to match the lower average yields in the data and the upward sloping yield curve. Intuitively, a lower long run mean leads to a lower level of nominal yields. Moreover, the lower perceived long run mean has a more significant impact on the short end than on the long end of the yield curve (the function $B(\tau)$ is strictly decreasing in maturity) and hence for a sufficiently low long run mean the yield curve is perceived as upward sloping by the econometrician.

We now ask how yield volatilities are perceived when the representative investor and the econometrician have different beliefs about the expected inflation mean reversion. Figure 4 shows that even if the model predicts that yield volatilities are too high an econometrician will perceive them as lower if he has a lower estimate for the variance of the short rate (a higher mean reversion coefficient). The figure plots the unconditional volatility of nominal bond yields as a function of the econometrician's mean reversion coefficient κ^0. Yield volatilities are flat (slightly downward sloping) in the data (Fama-Bliss data set). This is shown by the three dashed lines. The yield volatilities from the perspective of the econometrician are shown by the three solid lines. The green line represents the mean reversion

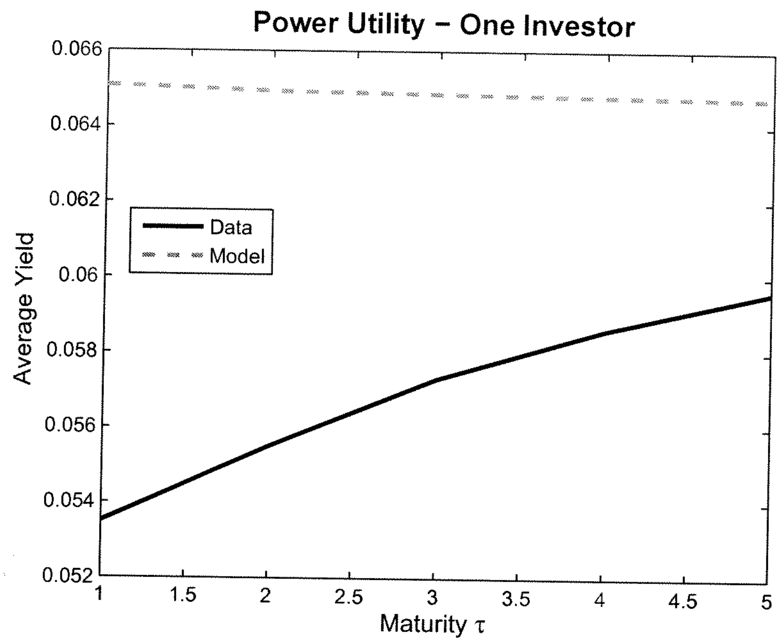

Figure 1: Average Yields

coefficient of the representative investor (RI) in the model (yields are determined with respect to his belief). Yield volatilities are downward sloping from the perspective of the RI as the intersection of the green line with the other three solid lines show. Yield volatilities predicted by the model (with respect to the beliefs of the RI) are significantly larger than in the data. If the econometrician's mean reversion coefficient is higher than that of the RI, then it is possible to match the lower yield volatilities in the data. Intuitively, less persistent expected inflation process (higher kappa) directly lowers the yield variance through the higher kappa and indirectly through the lower local volatility for the estimator of expected inflation σ_x^0. This effect is more pronounced for short term yields ($B_{P_\$}(\tau)$ is strictly decreasing in τ) and thus the term structure of yield volatilities is perceived as less downward sloping by the econometrician.

Finally, Figure 5 plots the volatility of the risk premium as a function of the econometrician's mean reversion coefficient κ^0. In particular, the plot highlights how differences in the econometrician's beliefs and the representative investor alone can generate volatility in the risk premium.

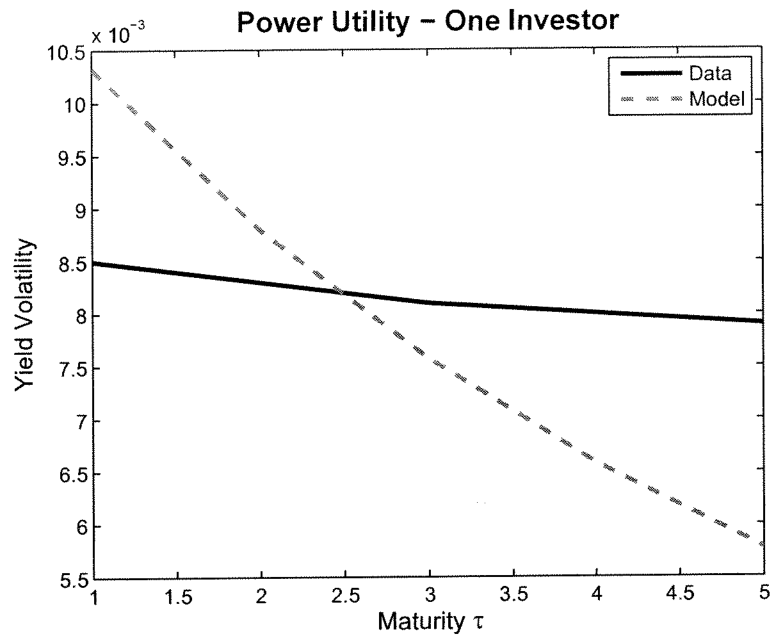

Figure 2: Yield Volatilities

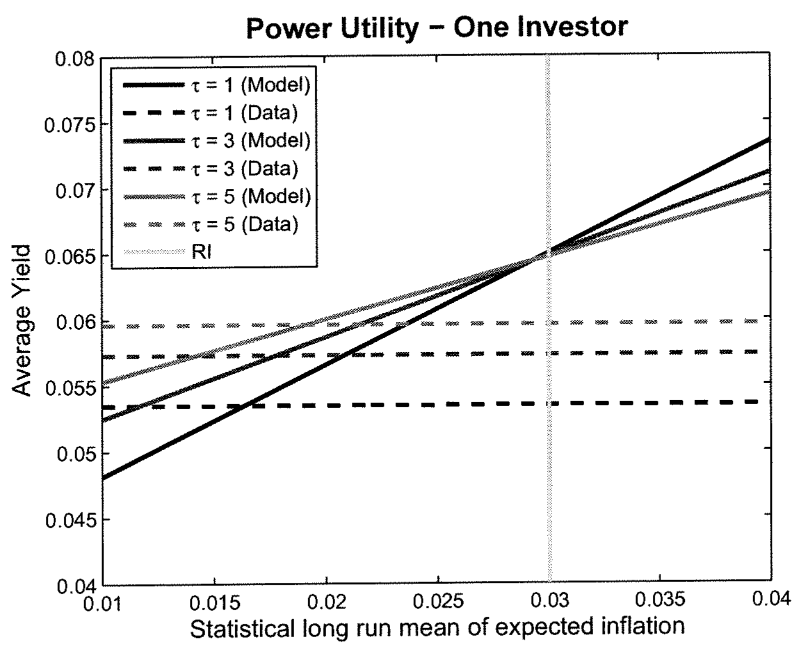

Figure 3: Average Yield Decomposition

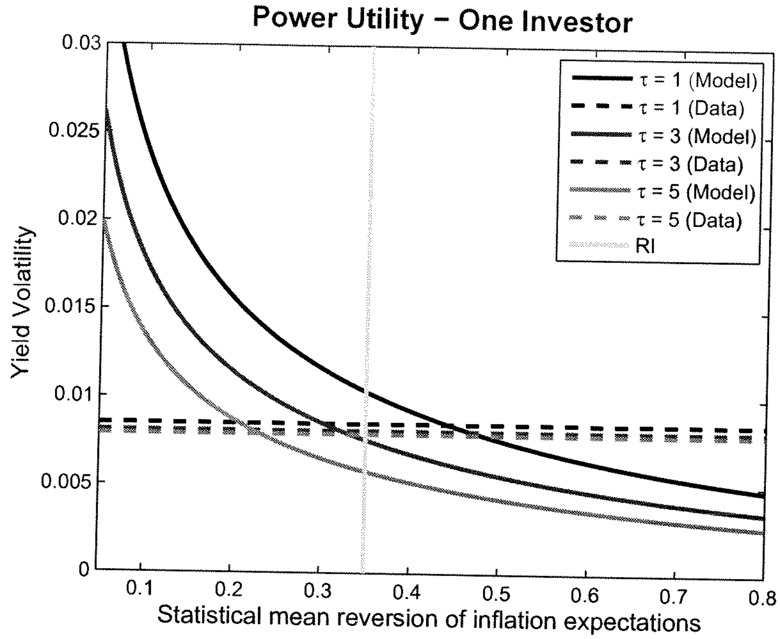

Figure 4: Yield Volatility Decomposition

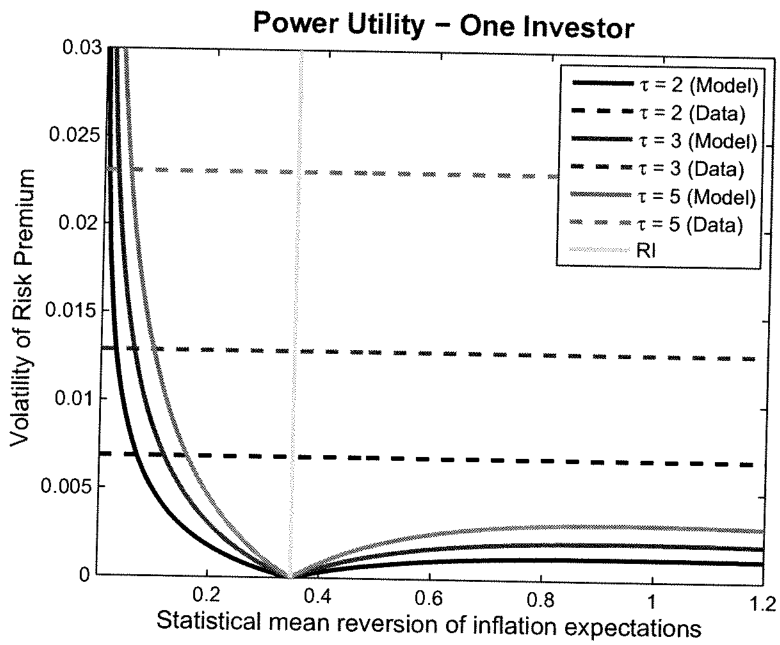

Figure 5: Risk Premium Decomposition

5 Conclusion

We study how heterogeneity in preferences and difference in beliefs about expected inflation influence the nominal term structure when investors have external habit formation preferences in a pure exchange continuous time economy. The main innovation of our works is to introduce differences in beliefs about expected inflation. In particular, investors observe the path of exogenous consumption and the exogenous price level, but have different beliefs about expected inflation. Differences in beliefs about expected inflation alone impacts the equilibrium real pricing kernel generating a spill-over effect from the nominal to the real side of the economy. The equilibrium nominal stochastic discount factor is determined in closed form and the effects of aggregate risk aversion, difference in beliefs, and inflation on the nominal short rate and the nominal market price of risk are explored. To develop intuition concerning the role of different beliefs between investors and an econometrician, we consider simplifying cases where the term structures can be computed in closed-form. We demonstrate the importance of the econometrician's beliefs on perceived bond properties as well as how the nature of the difference in beliefs about inflation among investors is important in generating predictability in asset prices and creating spill-over effects to the real side of the economy.

Appendix

Here we use the same notation as Ahn, Dittmar, and Gallant (2002).[4] Let $Y(t)$ denote a $N-$dimensional vector of state variables and $Z_M(t)$ a $M-$dimensional vector of independent Brownian motions.

Assumption 1. *The dynamics of the stochastic discount factor SDF(t) are*[5]

$$\frac{dSDF(t)}{SDF(t)} = -r(t)\,dt + 1_M'\;diag\left[\eta_{0m} + \eta_{Ym}'Y(t)\right]_M\,dZ_M(t) \tag{5.1}$$

with

$$\eta_0 = (\eta_{01}, \ldots, \eta_{0M})' \in \mathcal{R}^M \tag{5.2}$$

$$\eta_Y = (\eta_{Y1}, \ldots, \eta_{YM})' \in \mathcal{R}^{M \times N} \tag{5.3}$$

Hence, the market price of risk is an affine function of the state vector $Y(t)$.

Assumption 2. *The short rate is a quadratic function of the state variables:*

$$r(t) = \alpha + \beta' Y(t) + Y(t)'\Psi Y(t), \tag{5.4}$$

where α is a constant, β is an N-dimensional vector of constants, and Ψ is an $N \times N$ dimensional positive semidefinite matrix of constants.[6]

If the matrix Ψ is non singular, then $r(t) \geq \alpha - \frac{1}{2}\beta'\Psi^{-1}\beta\;\forall t$.

Assumption 3. *The state vector $Y(t)$ follows a multidimensional OU-process:*

$$dY(t) = (\mu + \xi Y(t))\,dt + \Sigma dZ_M(t), \tag{5.5}$$

where μ is an N-dimensional vector of constants, ξ is an N-dimensional square matrix of constants, and Σ is a $N \times M$-dimensional matrix of constants. We assume that ξ is diagonalizable and has

[4] In contrast to Ahn, Dittmar, and Gallant (2002): (i) we assume that the vector of Brownian motions driving the discount factor is identical to the vector of Brownian motions driving the state variables and thus Υ is the identify matrix, and (ii) we allow the vector of Brownian motions to have a dimension that is different from the number of state variables.

[5] An apostrophe denotes the transpose of a vector or matrix, $1_M'$ denotes a vector of ones, and diag $[Y_m']_M$ denotes an M-dimensional matrix with diagonal elements $(Y_1, \ldots, Y_m)$.

[6] We don't impose an additional parameter restriction that guarantees non-negativity of the short rate.

negative real components of eigenvalues. Specifically, $\xi = U\Lambda U^{-1}$ *in which* U *is the matrix of* N *eigenvectors and* Λ *is the diagonal matrix of eigenvalues.*

Let $V(t,\tau)$ denote the price of a zero-coupon bond and $y(t,\tau)$ the corresponding yield. Specifically,

$$V(t,\tau) = \mathrm{E}_t\left[\frac{\mathrm{SDF}(t+\tau)}{\mathrm{SDF}(t)}\right] \tag{5.6}$$

$$y(t,\tau) = -\frac{1}{\tau}\ln\left(V(t,\tau)\right). \tag{5.7}$$

The bond price and corresponding yield are given in the next proposition.

Proposition 8 (Quadratic Gaussian Term Structure Model). *Let* $\delta_0 = -\Sigma\Upsilon\eta_0 = -\Sigma\eta_0$ *and* $\delta_Y = -\Sigma\Upsilon\eta_Y = -\Sigma\eta_Y$. *The bond price is an exponential quadratic function of the state vector*

$$V(t,\tau) = exp\left\{A(\tau) + B(\tau)'Y(t) + Y(t)'C(\tau)Y(t)\right\}, \tag{5.8}$$

where $A(\tau)$, $B(\tau)$, *and* $C(\tau)$ *satisfy the ordinary differential equations,*

$$\frac{dC(\tau)}{d\tau} = 2C(\tau)\Sigma\Sigma'C(\tau) + \left(C(\tau)(\xi - \delta_Y) + (\xi - \delta_Y)'C(\tau)\right) - \Psi \tag{5.9}$$

$$\frac{dB(\tau)}{d\tau} = 2C(\tau)\Sigma\Sigma'B(\tau) + (\xi - \delta_Y)'B(\tau) + 2C(\tau)(\mu - \delta_0) - \beta \tag{5.10}$$

$$\frac{dA(\tau)}{d\tau} = trace\left[\Sigma\Sigma'C(\tau)\right] + \frac{1}{2}B(\tau)'\Sigma\Sigma'B(\tau) + B(\tau)'(\mu - \delta_0) - \alpha, \tag{5.11}$$

in which $A(0) = 0$, $B(0) = 0_N$, *and* $C(0) = 0_{N\times N}$. *Moreover, the yield is a quadratic function of the state vector* $Y(t)$*:*

$$y(t,\tau) = A_y(\tau) + B_y(\tau)'Y(t) + Y(t)'C_y(\tau)Y(t) \tag{5.12}$$

with $A_y(\tau) = -A(\tau)/\tau$, $B_y(\tau) = -B(\tau)/\tau$, *and* $C_y(\tau) = -C(\tau)/\tau$.

Proof. See Ahn, Dittmar, and Gallant (2002). □

If the short rate is an affine function of the state vector $Y(t)$, then the bond price is an exponential affine function of the state vector $Y(t)$ because $\Psi = 0_{N\times N}$ implies $C(\tau) = 0_{N\times N}$ for all τ. The bond price in this case belongs to the class of essential affine term structure models (see Duffee (2002)) and is given in the next corollary.

Proposition 9 (Essential Affine Term Structure Model). *Let $\Psi = 0_{N\times N}$, $\delta_0 = -\Sigma\Upsilon\eta_0 = -\Sigma\eta_0$ and $\delta_Y = -\Sigma\Upsilon\eta_Y = -\Sigma\eta_Y$ and assume that $(\xi - \delta_Y)$ is invertible. The bond price is an exponential affine function of the state vector*

$$V(t,\tau) = exp\left\{A(\tau) + B(\tau)'Y(t)\right\}, \tag{5.13}$$

where

$$B(\tau) = -\left((\xi-\delta_Y)'\right)^{-1}\left(e^{(\xi-\delta_Y)'\tau} - I_{N\times N}\right)\beta, \tag{5.14}$$

$I_{N\times N}$ denotes the N dimensional identity matrix, and

$$\begin{aligned}A(\tau) = &\frac{1}{2}\beta'\left(\int_0^\tau \left(e^{(\xi-\delta_Y)'u}\right)' K\, e^{(\xi-\delta_Y)'u}\, du\right)\beta \\ &- \left(\beta'K + (\mu-\delta_0)'\left((\xi-\delta_Y)'\right)^{-1}\right)\left(\int_0^\tau e^{(\xi-\delta_Y)'u}\, du\right)\beta \\ &+ \left(\frac{1}{2}\beta'K\,\beta + (\mu-\delta_0)'\left((\xi-\delta_Y)'\right)^{-1}\beta - \alpha\right)\tau\end{aligned} \tag{5.15}$$

with

$$K = \left(\left((\xi-\delta_Y)'\right)^{-1}\right)'\Sigma\Sigma'\left((\xi-\delta_Y)'\right)^{-1}. \tag{5.16}$$

If $(\xi-\delta_Y)'$ is diagonalizable; i.e. $(\xi-\delta_Y)' = T\Lambda T^{-1}$ then[7]

$$B(\tau) = -T diag\left[\frac{1}{\lambda_i}\left(e^{\lambda_i\tau} - 1\right)\right]T^{-1}\beta, \tag{5.17}$$

$$\int_0^\tau e^{(\xi-\delta_Y)'u}\, du = T diag\left[\frac{1}{\lambda_i}\left(e^{\lambda_i\tau} - 1\right)\right]T^{-1}, \tag{5.18}$$

and

$$\int_0^\tau \left(e^{(\xi-\delta_Y)'u}\right)' K\, e^{(\xi-\delta_Y)'u}\, du = \left(T^{-1}\right)' G(\Lambda, t)T^{-1}, \tag{5.19}$$

[7]The matrix $(\xi - \delta_Y)$ is invertible and thus all eigenvalues are nonzero.

where $G(\Lambda, t)$ *is a* $m \times m$*-matrix with elements given by*

$$G_{ij} = \frac{\omega_{ij}}{\lambda_i + \lambda_j}\left(e^{(\lambda_i+\lambda_j)t} - 1\right) \tag{5.20}$$

and ω_{ij} *denotes the element of the matrix* $\Omega = T'KT$ *in the* i^{th}*-row and* j^{th}*-column.*

Moreover, the yield is an affine function of the state vector $Y(t)$*:*

$$y(t, \tau) = A_y(\tau) + B_y(\tau)'Y(t) \tag{5.21}$$

with $A_y(\tau) = -A(\tau)/\tau$, *and* $B_y(\tau) = -B(\tau)/\tau$.

Proof. where $A(\tau)$ and $B(\tau)$ satisfy the ordinary differential equations,

$$\frac{dB(\tau)}{d\tau} = (\xi - \delta_Y)'B(\tau) - \beta \tag{5.22}$$

$$\frac{dA(\tau)}{d\tau} = \frac{1}{2}B(\tau)'\Sigma\Sigma'B(\tau) + B(\tau)'(\mu - \delta_0) - \alpha, \tag{5.23}$$

in which $A(0) = 0$ and $B(0) = 0_N$. $\square$

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